

# Benaloh's Dense Probabilistic Encryption Revisited<sup>\*</sup>

Laurent Fousse<sup>1</sup>, Pascal Lafourcade<sup>2</sup>, and Mohamed Alnuaimi<sup>3</sup>

<sup>1</sup> Université Grenoble 1

CNRS

Laboratoire Jean Kuntzmann

France

`Laurent.Fousse@imag.fr`

<sup>2</sup> Université Grenoble 1

CNRS

Verimag

France

`Pascal.Lafourcade@imag.fr`

<sup>3</sup> ENSIMAG

France

`alnuaimi.mohd@gmail.com`

**Abstract.** In 1994, Josh Benaloh proposed a probabilistic homomorphic encryption scheme, enhancing the poor expansion factor provided by Goldwasser and Micali's scheme. Since then, numerous papers have taken advantage of Benaloh's homomorphic encryption function, including voting schemes, computing multi-party trust privately, non-interactive verifiable secret sharing, online poker... In this paper we show that the original description of the scheme is incorrect, possibly resulting in ambiguous decryption of ciphertexts. We give a corrected description of the scheme and provide a complete proof of correctness. We also compute the probability of failure of the original scheme. Finally we analyze several applications using Benaloh's encryption scheme. We show in each case the impact of a bad choice in the key generation phase of Benaloh's scheme. For instance in the application of e-voting protocol, it can inverse the result of an election, which is a non negligible consequence.

**Keywords:** homomorphic encryption, public-key encryption, Benaloh's scheme.

## 1 Introduction

In the literature there are several homomorphic encryption schemes as for instance schemes proposed by Goldwasser-Micali [GM82], ElGamal [Elg85], Benaloh [Ben94], Naccache and Stern [NS98], Okamoto and Uchiyama [OU98], Paillier [Pai99] and its generalization proposed by Damgård and Jurik [DJ01],

---

<sup>\*</sup> This work was supported by ANR SeSur SCALP, SFINCS, AVOTE.

Sander, Young and Yung [SY99], Gaborit and Aguilar [MGH10]. In this paper we focus our attention on Benaloh's encryption scheme. In [CF07] a survey of existing homomorphic encryption schemes is proposed for the non specialist. In [Aki09] the author also proposes a description and a complexity analysis of different existing homomorphic encryption schemes. In [Rap06], the author considers homomorphic cryptosystems and their applications. In all these papers authors mention existing homomorphic encryption schemes and give descriptions of such schemes including Benaloh's scheme. Homomorphic encryption schemes have several applications. We only cite applications that are using Benaloh's scheme as for example voting schemes [BT94,RV05,CB87], computing multi-party trust privately [CDN01,JKM05,DGK10], non-interactive verifiable secret sharing [CB87], online poker [Gol05]...

Despite all these papers on applications and implementations realized by all these specialists, we were surprised to discover that the condition in the key generation of Benaloh's scheme can in some cases lead to an ambiguous encryption. How is it possible that after all these papers, results, protocols, even implementations and more than fifteen years nobody noticed it? In order to answer this question we will explicitly express the failure probability of the original scheme in Section 5. How did we discover this problem? We wanted to perform a time comparison of the efficiency of some well-known homomorphic encryptions. We proposed a methodology for testing their performance on large randomly generated data. So we started to code some of the encryption and decryption functions of homomorphic primitives. Benaloh's was one of the first one that we have tried. We were surprised to see that on some randomly generated instances of Benaloh's parameters our decryption function was not able to recover the correct plaintext. After verifying our code several times according to the conditions given in the original paper we were not able to understand why our code did not give the right plaintext. So we investigated more and were able to generate several counter-examples (one example of problematic parameters is given in Section 3) and more interestingly we clearly understood why and where the scheme failed. Indeed the bug is due to a very small detail, hence we proposed a revisited version of Benaloh's dense probabilistic encryption.

*Contributions:* The first contribution is that the original scheme proposed by Benaloh in [Ben94] does not give a unique decryption for all ciphertexts. We exhibit a simple example in the rest of the paper and characterize when this can happen and how to produce such counter-examples. Indeed the problem comes from the condition in the generation of the public key. The condition is not strong enough and allows to generate such keys that can for some plaintexts generate ambiguous ciphertexts.

Hence our second contribution is a new condition for the key generation which avoids such problem. We not only propose a new correct condition but also give an equivalent practical condition that can be used for implementations. We also compute the probability of failure of the original scheme, in order to justify why nobody discovered the problem before us.

Finally we describe some applications using explicitly Benaloh's scheme. In each case we briefly explain how the application works on a simple example. With these examples we clearly show that if our new condition is not used then the wrong key generation can have important consequences. In the case of the e-voting protocol it can change the result of an election; for private multi-party trust computation it can really impact the trust that somebody can have in someone.

*Outline:* In Section 2 we recall the original Benaloh scheme. In Section 3 we give a small example of parameters following the initial description and where we have ambiguous decryption. Then in Section 4 we give a corrected description of the scheme, with a proof of correctness. The probability of choosing incorrect parameters in the initial scheme is discussed in Section 5. In Section 6 we discuss some schemes related to Benaloh's scheme. Finally before concluding in the last section, we demonstrate using some applications that the problem we discover can have serious consequences.

## 2 Original Description of Benaloh's Scheme

Benaloh's "Dense Probabilistic Encryption" [Ben94] describes an homomorphic encryption scheme with a significant improvement in terms of expansion factor compared to Goldwasser-Micali [GM82]. For the same security parameter (the size of the RSA modulus  $n$ ), the ciphertext is in both cases an integer mod  $n$ , but the cleartext in Benaloh's scheme is an integer mod  $r$  for some parameter  $r$  depending on the key, whereas the cleartext in Goldwasser-Micali is only a bit. When computing the expansion factor for random keys, we found that it is most of the times close to  $1/2$  while it is  $\lceil \log_2(n) \rceil$  for Goldwasser-Micali. We now recall the three steps of the original scheme given in Benaloh's paper [Ben94].

*Key Generation* The public and private key are generated as follows:

- Choose a block size  $r$  and two large primes  $p$  and  $q$  such that :
  - $r$  divides  $(p - 1)$ .
  - $r$  and  $(p - 1)/r$  are relatively prime.
  - $r$  and  $q - 1$  are relatively prime.
  - $n = pq$ .
- Select  $y \in (\mathbb{Z}_n)^* = \{x \in \mathbb{Z}_n : \gcd(x, n) = 1\}$  such that

$$y^{\varphi/r} \not\equiv 1 \pmod{n} \tag{1}$$

where  $\varphi$  denotes  $(p - 1)(q - 1)$ .

The public key is  $(y, r, n)$ , and the private key is the two primes  $p, q$ .

*Encryption* If  $m$  is an element in  $\mathbb{Z}_r$  and  $u$  a random number in  $(\mathbb{Z}_n)^*$  then we compute the randomized encryption of  $m$  using the following formula:

$$E_r(m) = \{y^m u^r \pmod{n} : u \in (\mathbb{Z}_n)^*\}.$$

*Decryption* We first notice that for any  $m, u$  we have:

$$(y^m u^r)^{(p-1)(q-1)/r} \equiv y^{m(p-1)(q-1)/r} u^{(p-1)(q-1)} \equiv y^{m(p-1)(q-1)/r} \pmod{n}.$$

Since  $m < r$  and  $y^{(p-1)(q-1)/r} \not\equiv 1 \pmod{n}$ , Benaloh concludes that  $m = 0$  if and only if  $(y^m u^r)^{(p-1)(q-1)/r} \equiv 1 \pmod{n}$ . So if  $z = y^m u^r \pmod{n}$  is an encryption of  $m$ , given the secret key  $p, q$  we can determine whether  $m = 0$ . If  $r$  is small, we can decrypt  $z$  by doing an exhaustive search of the smallest non-negative integer  $m$  such that  $(y^{-m} z \pmod{n}) \in E_r(0)$ . By precomputing values and using the baby-step giant-step algorithm it is possible to perform the decryption in time  $O(\sqrt{r})$ . Finally if  $r$  is smooth we can use classical index-calculus techniques. More details about these optimization of decryption are discussed in the original paper [Ben94].

We remark that there is a balance to find between three parameters in this cryptosystem:

- ease of decryption, which requires that  $r$  is a product of small prime powers,
- a small expansion factor, defined as the ratio between the size of the ciphertexts and the size of the cleartexts. Because  $p$  and  $q$  have the same size and  $r \mid p-1$ , this expansion factor is at least 2,
- robustness of the private key, meaning that  $n$  should be hard to factor. In the context of the P-1 factorization method [Pol74], a big smooth factor of  $p-1$  is a definite weakness.

We notice that the cryptosystem of Naccache-Stern [NS98], similar to Benaloh's scheme, addresses this issue and by consequence do not produce ambiguous encryption.

### 3 A Small Counter-Example

We start by picking a secret key  $n = pq = 241 \times 179 = 43139$ , for which we can pick  $r = 15$ . Algorithm 1 may be used to compute the maximal suitable value of the  $r$  parameter if you start by picking  $p$  and  $q$  at random, but a smaller and smoother value may be used instead for an easier decryption.

---

**Algorithm 1** Compute  $r$  from  $p$  and  $q$ .

---

```

 $r \leftarrow p - 1$ 
while  $\gcd(q - 1, r) \neq 1$  do
     $r \leftarrow r / \gcd(r, q - 1)$ 
end while

```

---

We verify that  $r = 15$  divides  $p - 1 = 240 = 16 \times 15$ ,  $r$  and  $(p - 1)/r = 16$  are relatively prime,  $r = 15 = 3 \times 5$  and  $q - 1 = 178 = 2 \times 89$  are coprime. Assume we pick  $y = 27$ , then  $\gcd(y, n) = 1$  and  $y^{(p-1)(q-1)/r} = 40097 \not\equiv 1 \pmod{n}$  so

according to Benaloh's key generation procedure all the original conditions are satisfied.

By definition,  $z_1 = y^1 12^r = 24187$  is a valid encryption of  $m_1 = 1$ , while  $z_2 = y^6 4^r = 24187 = z_1$  is also a valid encryption of  $m_2 = 6$ . In fact we can verify that with this choice of  $y$ , the true cleartext space is now  $\mathbb{Z}_5$  instead of  $\mathbb{Z}_{15}$  (hence the ambiguity in decryption): first notice that in  $\mathbb{Z}_p$ ,  $y^5 = 8 = 41^{15}$ . This means that a valid encryption of 5 is also a valid encryption of 0. For any message  $m$ , the set of encryptions of  $m$  is the same as the set of encryptions of  $m + 5$ , hence the collapse in message space size. The fact that the message space size does not collapse further can be checked by brute force with this small set of parameters.

For this specific choice of  $p$  and  $q$ , there are  $\frac{r-1}{r}\varphi(n) = 39872$  possible values of  $y$  according to the original paper, but 17088 of them would lead to an ambiguity in decryption (that's a ratio of 3/7), sometimes decreasing the cleartext space to  $\mathbb{Z}_3$  or  $\mathbb{Z}_5$ . Details are provided in Section 5.

## 4 Corrected Version of Benaloh's Scheme

Let  $g$  be a generator of the group  $\mathbb{Z}_p^*$ , and since  $y$  is coprime with  $n$ , write  $y = g^\alpha \bmod p$ . We will now state in Theorem 1 our main contribution:

**Theorem 1** *The following properties are equivalent:*

- a)  $\alpha$  and  $r$  are coprime;
- b) decryption works unambiguously;
- c) For all prime factors  $s$  of  $r$ , we have  $y^{(\varphi/s)} \neq 1 \bmod n$ .

Of course property (b) is what we expect of the scheme, while (a) is useful to analyze the proportion of invalid  $y$ 's and (c) is more efficient to verify in practice than (a), especially considering that in order to decrypt efficiently the factorization of  $r$  is assumed to be known.

*Proof.* We start by showing (a)  $\Rightarrow$  (b). Assume two messages  $m_1$  and  $m_2$  are encrypted to the same element using nonce  $u_1$  and  $u_2$ :

$$y^{m_1} u_1^r = y^{m_2} u_2^r \bmod n.$$

Reducing mod  $p$  we get:

$$g^{\alpha(m_1 - m_2)} = (u_2/u_1)^r \bmod p$$

and using the fact that  $g$  is a generator of  $(\mathbb{Z}/p\mathbb{Z})^*$ , there exists some  $\beta$  such that

$$g^{\alpha(m_1 - m_2)} = g^{\beta r} \bmod p$$

which in turns implies

$$\alpha(m_1 - m_2) = \beta r \bmod p - 1.$$

By construction of  $r$ , we can further reduce mod  $r$  and get

$$\alpha(m_1 - m_2) = 0 \bmod r$$

and since  $r$  and  $\alpha$  are coprime, we can deduce  $m_1 = m_2 \bmod r$ , which means that decryption works unambiguously since the cleartexts are defined mod  $r$ .

We now prove that (a)  $\Rightarrow$  (c). Assume that there exists some prime factor  $s$  of  $r$  such that

$$y^{(\varphi/s)} = 1 \bmod n.$$

As above, by reducing mod  $p$  and using the generator  $g$  of  $(\mathbb{Z}/p\mathbb{Z})^*$  we get

$$\alpha \frac{\varphi}{s} = 0 \bmod p-1. \quad (2)$$

Let  $k = v_s(r)$  the  $s$ -valuation of  $p-1$  and write  $\alpha = \nu s + \mu$  the Euclidean division of  $\alpha$  by  $s$ . By construction we have  $v_s(\varphi) = k$ . When reducing (2) mod  $s^k$  we can remove all factors of  $\varphi$  that are coprime with  $s$ , so we get

$$\begin{aligned} \alpha s^{k-1} &= 0 \bmod s^k \\ \mu s^{k-1} &= 0 \bmod s^k \\ \mu &= 0 \bmod s \\ \mu &= 0 \end{aligned}$$

and  $\alpha$  and  $r$  are not coprime.

We now prove (c)  $\Rightarrow$  (a). Assume  $\alpha$  and  $r$  are not coprime and denote by  $s$  some common prime factor. Then

$$\begin{aligned} y^{(\varphi/s)} &= g^{\alpha\varphi/s} \bmod p \\ &= g^{(\alpha/s)\varphi} \bmod p = 1 \bmod p. \end{aligned}$$

And by construction of  $r$ ,  $s \nmid q-1$  so  $y^{(\varphi/s)} = 1 \bmod q$ .

We now prove (b)  $\Rightarrow$  (a). Assume two different cleartexts  $m_1 \neq m_2 \bmod r$  are encrypted to the same ciphertext using nonces  $u_1$  and  $u_2$ :

$$y^{m_1} u_1^r = y^{m_2} u_2^r \bmod n.$$

As before, we focus on operations mod  $p$  and we get

$$\alpha(m_1 - m_2) = 0 \bmod r.$$

If  $\alpha$  were invertible mod  $r$ , we would get an absurdity.

Notice than in the example of Section 3 we have  $y^{(p-1)(q-1)/3} = 1 \bmod n$  so condition (c) is not satisfied. We claimed that the real ciphertext space is now  $\mathbb{Z}_5$ , and we give a precise analysis of the cleartext space reduction at the end of Section 5.

## 5 Probability of Failure of Benaloh's Scheme

We now estimate the probability of failure in the scheme as originally described. For this we need to count the numbers  $y$  that satisfy condition (1) and not property (c) of Theorem 1. We call these values of  $y$  “faulty”.

**Lemma 1** *Condition (1) is equivalent to the statement:  $r \nmid \alpha$ .*

*Proof.* Assume that  $r$  divides  $\alpha$ :  $\alpha = r\alpha'$ . So

$$\begin{aligned} y^{\alpha/r} &= g^{\alpha\varphi/r} \bmod n \\ &= (g^{\alpha'})^\varphi \bmod n \\ &= 1 \bmod n. \end{aligned}$$

Conversely, if  $y^{\alpha/r} = 1 \bmod n$ , then

$$\begin{aligned} g^{\alpha\varphi/r} &= 1 \bmod n \\ &= 1 \bmod p \\ \alpha \frac{\varphi}{r} &= 0 \bmod p-1. \end{aligned}$$

Since  $r$  divides  $p-1$  and is coprime with  $\frac{\varphi}{r}$  (by definition), we have  $r \mid \alpha$ .

Since picking  $y \in (\mathbb{Z}_p)^*$  at random is the same when seen mod  $p$  as picking  $\alpha \in \{0, \dots, p-2\}$  at random, we can therefore conclude that the proportion  $\rho$  of faulty  $y$ 's is exactly the proportion of non-invertible numbers mod  $r$  among the non-zero mod  $r$ . So  $\rho = 1 - \frac{\varphi(r)}{r-1}$ . We notice that this proportion depends on  $r$  only, and it is non-zero when  $r$  is not a prime. Since decryption in Benaloh's scheme is essentially solving a discrete logarithm in a subgroup of  $\mathbb{Z}_p$  of order  $r$ , the original schemes recommends to use  $r$  as a product of small primes' powers, which tends to increase  $\rho$ . In fact, denoting by  $(p_i)$  the prime divisors of  $r$  we have:

$$\rho = 1 - \frac{r}{r-1} \prod_i \frac{p_i-1}{p_i} \approx 1 - \prod_i \frac{p_i-1}{p_i}$$

which shows that the situation where decryption is easy also increases the proportion of invalid  $y$  when using the initial description of the encryption scheme.

As a practical example, assume we pick two 512 bits primes  $p$  and  $q$  as

$$\begin{aligned} p &= 2 \times (3 \times 5 \times 7 \times 11 \times 13) \times p' + 1 \\ p' &= 4464804505475390309548459872862419622870251688508955 \backslash \\ &\quad 5037374496982090456310601222033972275385171173585381 \backslash \\ &\quad 3914691524677018107022404660225439441679953592 \\ q &= 1005585594745694782468051874865438459560952436544429 \backslash \\ &\quad 5033292671082791323022555160232601405723625177570767 \backslash \\ &\quad 523893639864538140315412108959927459825236754568279. \end{aligned}$$

Then

$$\begin{aligned}
\gcd(q-1, p-1) &= 2 \\
r &= (3 \times 5 \times 7 \times 11 \times 13) \times p' \\
\rho &= 1 - \frac{r}{r-1} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{10}{11} \times \frac{12}{13} \times \frac{p'}{p'-1} \\
\rho &> 61\%.
\end{aligned}$$

This example was constructed quite easily: first we take  $p'$  of suitable size, and increase its value until  $p$  is prime. Then we generate random primes  $q$  of suitable size until the condition  $\gcd(p-1, q-1) = 2$  is verified; it took less than a second on a current laptop using Sage [S<sup>+</sup>10].

Putting it all together, we can also characterize the faulty values of  $y$ , together with the actual value  $r'$  of the cleartext space size (compared to the expected value  $r$ ):

**Lemma 2** *Let  $u = \gcd(\alpha, r)$ . Then  $r' = \frac{r}{u}$ . Moreover if  $r' \neq r$ , this faulty value of  $y$  goes undetected by the initial condition as long as  $u \neq r$ .*

The proof of the first implication in Theorem 1 is easily extended to a proof of the first point of this lemma, while the second point is a mere rephrasing of the previous lemma.

This result can be used to craft counter-examples as we did in Section 3: for a valid value  $y$  of the parameter and  $u$  a proper divisor of  $r$ , the value  $y' = y^u \bmod n$  is an undetected faulty value with actual cleartext space size  $r' = r/u$ . It can also be used to determine precisely, for every proper divisor  $r'$  of  $r$  the probability of picking an undetected faulty parameter  $y$  of actual cleartext space size  $r'$ . Such an extensive study was not deemed necessary in the examples to follow in Section 7.

## 6 Related Schemes

We briefly discuss in this section some schemes related to that of [Ben94].

In [BT94], the authors describe a cryptosystem which closely resembles that of [Ben94], but the conditions given on  $r$  are less strict. Let us recall briefly the parameters of the cryptosystem as described in [BT94]:

- $r \mid p-1$  but  $r^2 \nmid p-1$ .
- $r \nmid q-1$ .
- $y$  is coprime with  $n$  and  $y^{(p-1)(q-1)/r} \neq 1 \bmod n$ .

It is clear that  $r^2 \nmid p-1$  is weaker than  $\gcd((p-1)/r, r) = 1$ , and that  $r \nmid q-1$  is weaker than  $\gcd(q-1, r) = 1$ . Therefore any set of parameters satisfying [Ben94] are also valid parameters as defined in [BT94].

Unfortunately the condition imposed on  $y$  is the same and still insufficient, and finding counter-examples is again a matter of picking  $\alpha$  not coprime with



$r$ . Our theorem still stands for this cryptosystem if you replace condition (c) by the following condition:

$$\text{For all prime factors } s \text{ of } r, \text{ we have } y^{(p-1)/s} \neq 1 \bmod p. \quad (3)$$

Going back in time, the scheme of Goldwasser and Micali [GM82] can be seen as a precursor of [BT94] with a fixed choice of  $r = 2$ . The choice of  $y$  in [GM82] as a quadratic non-residue mod  $n$  is clearly an equivalent formulation of condition (3).

Before [Ben94] and [BT94], the scheme was defined by Benaloh in [Ben87], with the parameter  $r$  being a prime. In this case our condition (c) is the same as the one proposed by Benaloh, and the scheme in this thesis is indeed correct. The main difference between the different versions proposed afterwards and this one is that it is not required for  $r$  to be prime, which leads in some cases to ambiguous ciphers. This remark clearly shows that all details are important in cryptography and that the problem we discover is subtle because even Benaloh himself did not notice it.

Finally the scheme proposed by Naccache and Stern [NS98] is quite close to the one proposed in [Ben87] but with a parameterization of  $p$  and  $q$ . It makes decryption correct, efficient, and leaves the expansion factor as an explicit function of the desired security level with respect to the  $P - 1$  method of factoring [Pol74] (the expansion is essentially the added size of the big cofactors of  $p - 1$  and  $q - 1$ ). We note in passing that a modulus size of 768 bits was considered secure at the time, a fact disproved twelve years later [KAF<sup>+</sup>10] only!

## 7 Applications

In this last section, we present some applications which explicitly use Benaloh's encryption scheme. We analyze in each situation what are the consequence on the application of using a bad parameter produced during the key generation.

### 7.1 Receipt-free Elections

In [BT94] the authors propose an application of homomorphic encryption for designing new receipt-free secret-ballot elections. They describe two protocols which use an homomorphic encryption verifying a list of properties. They also give in Appendix of the paper a precise description of a encryption scheme which satisfies their properties. Its relation with [Ben94] is given in section 6.

The new voting protocol uses the fact that the encryption is homomorphic and probabilistic. If we have two candidates Nicolas and Ségolène then the system associates for instance the ballot 0 for Nicolas and the ballot 1 for Ségolène. The main idea is that the server collects the  $m$  authenticated encrypted ballots  $\{v_i\}_k$  corresponding to the choices  $v_i$  of the  $m$  voters. Hence the server performs the multiplication of all these votes and decrypts the product once to obtain the result. The number obtained corresponds to the number of votes for Ségolène  $n_S$  and the difference  $m - n_S$  gives the number of votes for Nicolas.

We construct a basic application of the first protocol proposed in [BT94] and based on the example described in Section 3. In this example we consider only 20 voters. If the encryption is correctly done then the final result is  $\{14\}_k$ . It means that after decryption Ségolène has 14 votes and Nicolas has 6 votes. But if as we explain in Section 3 instead of computing the result  $14 \bmod 15$  we are taking the result modulo 5, then we obtain a result of  $14 \bmod 5 = 4$ . This time Nicolas obtains 16 votes and Ségolène only 4. This example clearly shows that this bug in the condition in the original paper can have important consequences.

## 7.2 Private Multi-Party Trust Computation

In [DGK10] the authors give a multiple private keys protocol for private multi-party computation of a trust value: an initiating user wants to know the (possibly weighted) average trust the network of nodes has in some user. In a first phase of the protocol, each of the  $n$  nodes splits its trust  $t$  in  $n - 1$  shares ( $s_i$ ) such that

$$t = s_1 + s_2 + \dots + s_{n-1} \bmod r.$$

Here  $r$  is a common modulus chosen big enough with respect to the maximum possible global trust value, and in order to insure the privacy of its trust value the shares should be taken as random number mod  $r$ , except for the last one. The shares are then sent encrypted (using Benaloh’s scheme) to each other user, to be later recombined. If we assume that one of the users has chosen a faulty value for his public parameter  $y$ , then his contribution to the recombined value will be computed mod  $r'$  instead of mod  $r$  for some divisor  $r'$  of  $r$ . As an extreme example, assume

- that the queried user is a newcomer, untrusted by anyone (hence the private value of  $t$  for every node is 0),
- that the true recombined value contributed by the faulty user should have been  $r - 1$ ,
- that  $r' = r/3$ .

Due to his miscalculation, the faulty node will contribute the value  $r' - 1$  instead of  $-1$ , causing the apparent calculated trust value to be quite high (about  $1/3$  of the maximum possible trust value, instead of 0). This can have dramatic consequences if the trust value is used later on to grant access to some resource. These assumptions are not entirely unlikely: remember that  $r = 3^k$  is an explicitly suggested choice of parameter of the cryptosystem (chosen for instance in [JKM05]) in which case  $\rho$  is close to  $1/3$  and faulty nodes occur with high probability even with moderate-sized networks. We note also that the description from [Ben94] is given *in extenso*, with its incorrect condition. One reason for choosing Benaloh’s cryptosystem in this application is because the cleartext space can be common among several private keys, a feature unfortunately not achieved *e.g.* by Paillier’s cryptosystem [Pai99] but also possible with Naccache-Stern’s [NS98].

### 7.3 Secure Cards Dealing

Another application of this encryption scheme is given in [Gol05]: securely dealing cards in poker (or similar games). Here again the author gives the complete description of the original scheme, with a choice of parameter  $r = 53$  (which is prime). Because  $r$  is prime, this application does not suffer from the flaw explained here, but this choice of a prime number is done for reasons purely internal to the cards dealing protocol, namely testing the equality of dealt cards.

Given two ciphertext  $E(m_1)$  and  $E(m_2)$ , the players need to test if  $m_1 = m_2$  without revealing anything more about the cards  $m_1$  and  $m_2$ . The protocol is as follows:

1. Let  $m = m_1 - m_2$ , each player can compute  $E(m) = E(m_1)/E(m_2)$  because of the homomorphic property of the encryption.
2. Each player  $P_i$  secretly picks a value  $0 < \alpha_i < 53$ , computes  $E(m)^{\alpha_i}$  and discloses it to everyone.
3. Each player can compute  $\prod_i E(m)^{\alpha_i} = E(m)^\alpha$  with  $\alpha = \sum_i \alpha_i$ . The players jointly decrypt  $E(m)^\alpha$  to get the value  $m\alpha \bmod r$ .

Now because for each player the value of  $\alpha$  is unknown and random, if  $m\alpha \neq 0 \bmod r$  then the players learn nothing about  $m$ . Otherwise they conclude that the cards are equal.

We claim that this protocol fails to account for two problems:

- there is no guarantee that  $\alpha \neq 0 \bmod r$ . When this happens, two distinct cards will be incorrectly considered equal.
- knowing the value of  $E(m)$  and  $E(m)^{\alpha_i}$ , it is easy to recover  $\alpha_i$  because of the small search space for  $\alpha_i$ . This means the protocol leaks information when  $m_1 \neq m_2$ . The fix here is to multiply by some random encryption of 0.

## 8 Conclusion

We have shown that the original definition of Benaloh's homomorphic encryption does not give sufficient conditions in the choice of public key to get an unambiguous encryption scheme. We gave a necessary and sufficient condition which fixes the scheme. Our discussion on the probability of choosing an incorrect public key shows that this probability is non negligible for parameters where decryption is efficient: for example using the suggested value of the form  $r = 3^k$ , this probability is already close to  $1/3$ . We also explain on some examples what can be the consequences of the use of the original Benaloh scheme. In fact, it is surprising this result was not found before, considering the number of applications built on the homomorphic property of Benaloh's scheme. This strongly suggests this scheme was rarely implemented.

## References

- Aki09. M. Akinwande. Advances in Homomorphic Cryptosystems. *Journal of Universal Computer Science*, 15(3):506–522, 2009.
- Ben87. Josh Daniel Cohen Benaloh. *Verifiable Secret-Ballot Elections*. PhD thesis, New Haven, CT, USA, 1987.
- Ben94. Josh Benaloh. Dense Probabilistic Encryption. In *In Proceedings of the Workshop on Selected Areas of Cryptography*, pages 120–128, 1994.
- BT94. Josh Benaloh and Dwight Tuinstra. Receipt-free Secret-Ballot Elections (extended abstract). In *STOC '94: Proceedings of the twenty-sixth annual ACM symposium on Theory of computing*, pages 544–553, New York, NY, USA, 1994. ACM.
- CB87. Josh Cohen Benaloh. Secret Sharing Homomorphisms: Keeping Shares of a Secret Secret. In *Proceedings on Advances in Cryptology—CRYPTO '86*, pages 251–260, London, UK, 1987. Springer-Verlag.
- CDN01. Ronald Cramer, Ivan Damgård, and Jesper Buus Nielsen. Multiparty Computation from Threshold Homomorphic Encryption. In *EUROCRYPT '01: Proceedings of the International Conference on the Theory and Application of Cryptographic Techniques*, pages 280–299, London, UK, 2001. Springer-Verlag.
- CF07. Caroline Fontaine and Fabien Galand. A Survey of Homomorphic Encryption for Nonspecialists. In *EURASIP Journal on Information Security*. Hindawi Publishing Corporation, 2007.
- DGK10. Shlomi Dolev, Niv Gilboa, and Marina Kopeetsky. Computing Multi-Party Trust Privately: in  $O(n)$  time units sending one (possibly large) message at a time. In *SAC '10: Proceedings of the 2010 ACM Symposium on Applied Computing*, pages 1460–1465, New York, NY, USA, 2010. ACM.
- DJ01. I. Damgård and M. Jurik. A Generalisation, a Simplification and Some Applications of Paillier’s Probabilistic Public-Key System. In *Public Key Cryptography*, page 119–136, 2001.
- Elg85. Taher Elgamal. Proceedings of CRYPTO 84 on Advances in Cryptology. In *A Public key Cryptosystem and a Signature Scheme Based on Discrete Logarithms*, pages 10–18. Springer-Verlag New York, Inc., Santa Barbara, California, United States, 1985.
- GM82. Shafi Goldwasser and Silvio Micali. Probabilistic Encryption and How to Play Mental Poker Keeping Secret All Partial Information. In *STOC*, pages 365–377, 1982.
- Gol05. Philippe Golle. Dealing Cards in Poker Games. In *Proc. of ITCC 2005 E-Gaming Track*, 2005.
- JKM05. Somesh Jha, Luis Kruger, and Patrick McDaniel. Privacy Preserving Clustering. In Sabrina de Capitani di Vimercati, Paul Syverson, and Dieter Gollmann, editors, *Computer Security – ESORICS 2005*, volume 3679 of *Lecture Notes in Computer Science*, pages 397–417. Springer Berlin / Heidelberg, 2005.
- KAF<sup>+</sup>10. Thorsten Kleinjung, Kazumaro Aoki, Jens Franke, Arjen K. Lenstra, Emmanuel Thomé, Pierrick Gaudry, Peter L. Montgomery, Dag Arne Osvik, Herman Te Riele, Andrey Timofeev, and Paul Zimmermann. Factorization of a 768-bit RSA modulus, 2010.
- MGH10. Carlos Aguilar Melchor, Philippe Gaborit, and Javier Herranz. Additively Homomorphic Encryption with  $d$ -Operand Multiplications. In Tal Rabin,

- editor, *CRYPTO*, volume 6223 of *Lecture Notes in Computer Science*, pages 138–154. Springer, 2010.
- NS98. David Naccache and Jacques Stern. A New Public Key Cryptosystem Based on Higher Residues. In *ACM Conference on Computer and Communications Security*, pages 59–66, 1998.
- OU98. T. Okamoto and S. Uchiyama. A New Public-key Cryptosystem as Secure as Factoring. In *Proc. International Conference on the Theory and Application of Cryptographic Techniques (EUROCRYPT’98)*, volume 1403, pages 308–318, Helsinki, Finland, 1998. Springer-Verlag.
- Pai99. P. Paillier. Public-Key Cryptosystems Based on Composite Degree Residuosity Classes. In *Proc. International Conference on the Theory and Application of Cryptographic Techniques (EUROCRYPT’99)*, volume 1592, pages 223–238, Prague, Czech Republic, 1999. Springer-Verlag.
- Pol74. J. M. Pollard. Theorems on Factorization and Primality Testing. *Mathematical Proceedings of the Cambridge Philosophical Society*, 76(03):521–528, 1974.
- Rap06. Doerte K. Rappe. Homomorphic Cryptosystems and their Applications. Cryptology ePrint Archive, Report 2006/001, 2006. <http://eprint.iacr.org/>.
- RV05. Alexandre Ruiz and Jorge Luis Villar. Publicly Verifiable Secret Sharing from Paillier’s Cryptosystem. In Christopher Wolf, Stefan Lucks, and Po-Wah Yau, editors, *WEWoRC*, volume 74 of *LNI*, pages 98–108. GI, 2005.
- S<sup>+</sup>10. W. A. Stein et al. *Sage Mathematics Software (Version 4.5.1)*. The Sage Development Team, 2010. <http://www.sagemath.org>.
- SY99. Tomas Sander, Adam Young, and Moti Yung. Non-Interactive CryptoComputing for NC<sup>1</sup>. In *FOCS*, pages 554–567, 1999.